

AREA-MINIMIZING CONES OVER MINIMAL EMBEDDINGS OF R-SPACES

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ABSTRACT. In this paper, by constructing area-nonincreasing retractions, we prove area-minimizing properties of some cones over minimal embeddings of R-spaces.

1. INTRODUCTION

Let C_B be the cone over a submanifold B of the unit sphere S^{n-1} in \mathbb{R}^n . The cone C_B is minimal in \mathbb{R}^n if and only if B is minimal in S^{n-1} . We call a cone C_B area-minimizing if its truncated cone C_B^1 has the least area among all integral currents with the same boundary B . Solutions of Plateau's problem can have singularities as integral currents. At an isolated conical singularity, the tangent cone is area-minimizing. Hence, in order to understand such singularities, we should study area-minimizing properties of minimal cones.

Lawlor [L] gave a sufficient condition, so-called the *curvature criterion*, for a cone to be area-minimizing, using an area-nonincreasing retraction. With this criterion, he obtained a complete classification of area-minimizing cones over products of spheres and the first examples of area-minimizing cones over nonorientable manifolds. Kerckhove [Ke] proved that some cones over isolated orbits of the adjoint representations of $SU(n)$ and $SO(n)$ are area-minimizing. A symmetric R-space can be minimally embedded in the sphere in a canonical way. Hirohashi, Kanno and Tasaki [HKT] constructed area-nonincreasing retractions onto the cones over symmetric R-spaces associated with symmetric pairs of type B_l . Furthermore, Kanno [Ka] proved that cones over some symmetric R-spaces are area-minimizing.

In this paper, we study area-minimizing properties of cones over minimal embeddings of R-spaces, not only symmetric R-spaces. In Theorem 2, we give a construction of retractions generalizing the method given in [HKT]. Applying this theorem we give some examples of area-minimizing cones over minimal embeddings of R-spaces. In Section 5, we discuss area-minimizing properties of cones over products of R-spaces.

2. PRELIMINARIES

2.1. Area-minimizing cones. Let B be a submanifold of the unit sphere S^{n-1} in \mathbb{R}^n . We define the cone C_B and the truncated cone C_B^1 over B by

$$\begin{aligned} C_B &= \{tx \in \mathbb{R}^n \mid 0 \leq t, x \in B\}, \\ C_B^1 &= \{tx \in \mathbb{R}^n \mid 0 \leq t \leq 1, x \in B\}. \end{aligned}$$

Both C_B and C_B^1 have an isolated singularity at the origin $0 \in \mathbb{R}^n$.

Definition 1. A cone C_B is called area-minimizing if C_B^1 has the least area among all integral currents with boundary B .

Let V and W be two vector spaces with inner products, and let $F : V \rightarrow W$ be a linear map. Suppose $\dim V = n \geq \dim W = m$. We define the Jacobian JF of F by

$$JF = \sup\{\|F(v_1) \wedge \cdots \wedge F(v_m)\|\},$$

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where $\{v_1, \dots, v_m\}$ runs over all orthonormal systems of V . If F is not surjective, then $JF = 0$. If F is surjective, then

$$JF = \|F(v_1) \wedge \dots \wedge F(v_m)\|$$

for any orthonormal basis of $(\ker F)^\perp$.

Definition 2. A retraction $\Phi : \mathbb{R}^n \rightarrow C_B$ is called differentiable if $\Phi : \mathbb{R}^n \setminus \Phi^{-1}(0) \rightarrow C_B \setminus \{0\}$ is C^1 . A differentiable retraction Φ is called area-nonincreasing if $J(d\Phi)_x \leq 1$ holds for all $x \in \mathbb{R}^n \setminus \Phi^{-1}(0)$.

Proposition 1. Let B be a compact submanifold of the unit sphere S^{n-1} in \mathbb{R}^n . Suppose that there exists an area-nonincreasing retraction Φ from \mathbb{R}^n to C_B . Then C_B is area-minimizing.

Proof. Let S be an integral current which has the same boundary B as C_B^1 . Since $\Phi(S) \supset C_B^1$, we have

$$\begin{aligned} \text{Vol}(C_B^1) &\leq \text{Vol}(\Phi(S)) = \text{Vol}(\Phi(S) \setminus \{0\}) \leq \int_{S \setminus \Phi^{-1}(0)} \|d\Phi(e_1 \wedge \dots \wedge e_k)\| d\mu_S \\ &\leq \int_{S \setminus \Phi^{-1}(0)} J(d\Phi)_x d\mu_S \leq \int_{S \setminus \Phi^{-1}(0)} 1 d\mu_S \leq \int_S 1 d\mu_S = \text{Vol}(S), \end{aligned}$$

where $\{e_1, \dots, e_k\}$ is an orthonormal frame of S . \square

If C_B is area-minimizing, then C_B is minimal in \mathbb{R}^n . Therefore, to find area-minimizing cones, it suffices to consider cones over minimal submanifolds of S^{n-1} . For this purpose, we use s -representations, which are the linear isotropy representations of Riemannian symmetric spaces.

2.2. Riemannian symmetric pairs and restricted root systems. Let G be a connected Lie group and θ be an involutive automorphism of G . We denote by $F(\theta, G)$ the fixed point set of θ , and we denote by $F(\theta, G)_0$ the identity component of $F(\theta, G)$. For a closed subgroup K of G , the pair (G, K) is said to be a Riemannian symmetric pair if $F(\theta, G)_0 \subset K \subset F(\theta, G)$ and $\text{Ad}(K)$ is compact. Let (G, K) be a Riemannian symmetric pair, and \mathfrak{g} and \mathfrak{k} be Lie algebras of G and K , respectively. We immediately see that

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid d\theta(X) = X\}.$$

We put

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid d\theta(X) = -X\}.$$

We denote by $\langle \cdot, \cdot \rangle$ an inner product on \mathfrak{g} which is invariant under the actions of $\text{Ad}(K)$ and $d\theta$. Then $\langle \cdot, \cdot \rangle$ induces a left-invariant metric on G and a G -invariant metric on $M = G/K$ to be a Riemannian symmetric space, which are denoted by the same symbol $\langle \cdot, \cdot \rangle$. Since $d\theta$ is involutive, we have an orthogonal direct sum decomposition of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

This decomposition is called the canonical decomposition of $(\mathfrak{g}, \mathfrak{k})$. For the origin $o \in G/K$, we can identify the tangent space $T_o(G/K)$ with \mathfrak{m} by the differential of the natural projection $\pi : G \rightarrow G/K$.

In this paper, we consider only Riemannian symmetric spaces of compact type. We suppose that G is compact and semisimple. Take and fix a maximal abelian subspace \mathfrak{a} in \mathfrak{m} and a maximal abelian subalgebra \mathfrak{t} in \mathfrak{g} including \mathfrak{a} . For $\lambda \in \mathfrak{t}$, we put

$$\tilde{\mathfrak{g}}_\lambda = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \sqrt{-1}\langle \lambda, H \rangle X \ (H \in \mathfrak{t})\}$$

and define the root system \tilde{R} of \mathfrak{g} by

$$\tilde{R} = \{\lambda \in \mathfrak{t} \setminus \{0\} \mid \tilde{\mathfrak{g}}_\lambda \neq \{0\}\}.$$

For $\lambda \in \mathfrak{a}$, we put

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \sqrt{-1}\langle \lambda, H \rangle X \ (H \in \mathfrak{a})\}$$

and define the restricted root system R of $(\mathfrak{g}, \mathfrak{k})$ by

$$R = \{\lambda \in \mathfrak{a} \setminus \{0\} \mid \mathfrak{g}_\lambda \neq \{0\}\}.$$

Denote the orthogonal projection from \mathfrak{t} to \mathfrak{a} by $H \mapsto \overline{H}$. We extend a basis of \mathfrak{a} to that of \mathfrak{t} and define a lexicographic orderings $>$ on \mathfrak{a} and \mathfrak{t} with respect to these basis. Then for $H \in \mathfrak{t}$, $\overline{H} > 0$ implies $H > 0$. We denote by \tilde{R} the fundamental system of \tilde{R} with respect to $>$, by F the fundamental system of R with respect to $>$. We define

$$\tilde{R}_0 = \{\lambda \in \tilde{R} \mid \overline{\lambda} = 0\}, \quad \tilde{F}_0 = \{\alpha \in \tilde{F} \mid \overline{\alpha} = 0\}.$$

Then we have

$$R = \{\overline{\lambda} \mid \lambda \in \tilde{R} \setminus \tilde{R}_0\}, \quad F = \{\overline{\alpha} \mid \alpha \in \tilde{F} \setminus \tilde{F}_0\}.$$

We denote the set of positive roots by

$$\tilde{R}_+ = \{\lambda \in \tilde{R} \mid \lambda > 0\}, \quad R_+ = \{\lambda \in R \mid \lambda > 0\}.$$

We put

$$\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [H, X] = 0 \ (H \in \mathfrak{a})\}$$

and for each $\lambda \in R_+$

$$\mathfrak{k}_\lambda = \mathfrak{k} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}), \quad \mathfrak{m}_\lambda = \mathfrak{m} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}).$$

We then have the following lemma.

Lemma 1 ([T]). (1) *We have orthogonal direct sum decompositions:*

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in R_+} \mathfrak{k}_\lambda, \quad \mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in R_+} \mathfrak{m}_\lambda.$$

(2) *For each $\mu \in \tilde{R}_+ \setminus \tilde{R}_0$ there exist $S_\mu \in \mathfrak{k}$ and $T_\mu \in \mathfrak{m}$ such that*

$$\{S_\mu \mid \mu \in \tilde{R}_+, \overline{\mu} = \lambda\}, \quad \{T_\mu \mid \mu \in \tilde{R}_+, \overline{\mu} = \lambda\}$$

are, respectively, orthonormal bases of \mathfrak{k}_λ and \mathfrak{m}_λ and that for any $H \in \mathfrak{a}$

$$[H, S_\mu] = \langle \mu, H \rangle T_\mu, \quad [H, T_\mu] = -\langle \mu, H \rangle S_\mu.$$

For each $\lambda \in R_+$ we put $m(\lambda) = \dim \mathfrak{m}_\lambda = \dim \mathfrak{k}_\lambda$. $m(\lambda)$ is called the multiplicity of λ . We define a subset D of \mathfrak{a} by

$$D = \bigcup_{\lambda \in R_+} \{H \in \mathfrak{a} \mid \langle \lambda, H \rangle = 0\}.$$

Each connected component of $\mathfrak{a} \setminus D$ is called a Weyl chamber. We define the fundamental Weyl chamber \mathcal{C} by

$$\mathcal{C} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle > 0 \ (\alpha \in F)\}.$$

The closure of \mathcal{C} is given by

$$\overline{\mathcal{C}} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle \geq 0 \ (\alpha \in F)\}.$$

For each subset $\Delta \subset F$, we define a subset $\mathcal{C}^\Delta \subset \overline{\mathcal{C}}$ by

$$\mathcal{C}^\Delta = \{H \in \overline{\mathcal{C}} \mid \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \langle \beta, H \rangle = 0 \ (\beta \in F \setminus \Delta)\}.$$

Then we have the following lemma.

Lemma 2 ([HKT]). (1) *For $\Delta_1 \subset F$*

$$\overline{\mathcal{C}^{\Delta_1}} = \bigcup_{\Delta \subset \Delta_1} \mathcal{C}^\Delta$$

is a disjoint union. In particular $\overline{\mathcal{C}} = \bigcup_{\Delta \subset F} \mathcal{C}^\Delta$ is a disjoint union.

(2) $\Delta_1 \subset \Delta_2$ if and only if $\mathcal{C}^{\Delta_1} \subset \overline{\mathcal{C}^{\Delta_2}}$, for $\Delta_1, \Delta_2 \subset F$.

For each $\alpha \in F$ we define $H_\alpha \in \mathfrak{a}$ by

$$\langle H_\alpha, \beta \rangle = \delta_{\alpha\beta} \quad (\beta \in F),$$

where $\delta_{\alpha\beta}$ is Kronecker's delta. Then for $\Delta \subset F$ we have

$$\mathcal{C}^\Delta = \left\{ \sum_{\alpha \in \Delta} x_\alpha H_\alpha \mid x_\alpha > 0 \right\}.$$

3. CONSTRUCTION OF RETRACTIONS

The notation of the preceding section will be preserved. The linear isotropy representation of a Riemannian symmetric space G/K is called an s -representation. The s -representation of G/K on $T_o(G/K)$ and the adjoint representation $\text{Ad}(K)$ on \mathfrak{m} are equivalent. Since an s -representation is an orthogonal representation, for a unit vector $H \in \mathfrak{m}$, the orbit $\text{Ad}(K)H$ is a submanifold of the unit sphere $S \subset \mathfrak{m}$. Orbits of s -representations are called R-spaces. The orbit space of an s -representation is homeomorphic to $\overline{\mathcal{C}}$, more precisely for any $X \in \mathfrak{m}$, there exists $k \in K$ and unique $H \in \overline{\mathcal{C}}$ such that $X = \text{Ad}(k)H$. The decomposition of $\overline{\mathcal{C}}$ in Lemma 2 is the decomposition of the orbit type. From the following theorem, we can see that for each orbit type, there exists a unique minimal orbit.

Theorem 1 ([HTST]). *For any nonempty subset $\Delta \subset F$, there exists a unique $H \in S \cap \mathcal{C}^\Delta$ such that the linear isotropy orbit $\text{Ad}(K)H$ is a minimal orbit of S .*

Corollary 1. *An isolated orbit (i.e. $\Delta = \{\alpha\}$) is a minimal submanifold of S .*

Kitagawa and Ohnita ([KO]) calculated the mean curvature vector m_H of $\text{Ad}(K)H$ in \mathfrak{m} at H :

$$m_H = - \sum_{\lambda \in \tilde{R}_+ \setminus \tilde{R}_+^\Delta} \frac{\bar{\lambda}}{\langle \lambda, H \rangle}.$$

This expression is used in the proof of Theorem 1. We consider cones over minimal embeddings of R-spaces that obtained in this way, and construct retractions.

Lemma 3 ([HKT]). *Suppose ϕ is a mapping of $\overline{\mathcal{C}}$ into itself such that $\phi(\mathcal{C}^\Delta) \subset \overline{\mathcal{C}^\Delta}$ for each $\Delta \subset F$. Then ϕ extends to a mapping Φ of \mathfrak{m} as*

$$\Phi(X) = \text{Ad}(k)\phi(H)$$

for each $X = \text{Ad}(k)H$ ($k \in K, H \in \overline{\mathcal{C}}$).

The following theorem is a generalization of Proposition 2.6 in [HKT].

Theorem 2. *For $A \in \overline{\mathcal{C}}$, we put $\Delta_0 = \{\alpha \in F \mid \langle \alpha, A \rangle > 0\}$. Let $f : \overline{\mathcal{C}} \rightarrow \mathbb{R}_{\geq 0}$ be a continuous function. Define a continuous mapping $\phi : \overline{\mathcal{C}} \rightarrow \{tA \mid t \geq 0\}$ by $\phi(x) = f(x)A$. If f satisfies*

- (1) $f(tA) = t$ ($t \geq 0$),
- (2) $f|_{\mathcal{C}^\Delta} = 0$ ($\Delta \subset F$ with $\Delta_0 \not\subset \Delta$),

then ϕ extends to a retraction $\Phi : \mathfrak{m} \rightarrow C_{\text{Ad}(K)A}$.

Proof. First, we show that ϕ satisfies the assumption of Lemma 3. For $\Delta \subset F$ if $\Delta_0 \subset \Delta$, then $\mathcal{C}^{\Delta_0} \subset \overline{\mathcal{C}^\Delta}$. Hence

$$\phi(\mathcal{C}^\Delta) = \{tA \mid t \geq 0\} \subset \mathcal{C}^{\Delta_0} \subset \overline{\mathcal{C}^\Delta}$$

holds. If $\Delta_0 \not\subset \Delta$, then $\phi(\mathcal{C}^\Delta) = \{0\}$ since $f|_{\mathcal{C}^\Delta} = 0$. Therefore, ϕ satisfies the assumption of Lemma 3. We also get

$$\begin{aligned} \Phi(\mathfrak{m}) &= \{\text{Ad}(k)f(H)A \mid k \in K, H \in \overline{\mathcal{C}}\} \\ &= \{t\text{Ad}(k)A \mid k \in K, t \geq 0\} = C_{\text{Ad}(K)A}. \end{aligned}$$

Thus Φ is a surjection from \mathfrak{m} onto $C_{\text{Ad}(K)A}$. Next we show that Φ is continuous. Let $\{P_n\}_{n \in \mathbb{N}}$ be a sequence in \mathfrak{m} with limit $P_\infty \in \mathfrak{m}$. Points P_n and P_∞ can be expressed as

$P_n = \text{Ad}(k_n)H_n$, $P_\infty = \text{Ad}(k_\infty)H_\infty$ where $k_n, k_\infty \in K$ and $H_n, H_\infty \in \overline{\mathcal{C}}$. Since the projection $\mathfrak{m} \rightarrow \overline{\mathcal{C}}; X = \text{Ad}(k)H \mapsto H$ is continuous, we have $\lim_{n \rightarrow \infty} H_n = H_\infty$. We put $\Delta_\infty = \{\alpha \in F \mid \langle \alpha, H_\infty \rangle > 0\}$, $Z_K^{H_\infty} = \{k \in K \mid \text{Ad}(k)H_\infty = H_\infty\}$ and $Z_K^{\Delta_\infty} = \{k \in K \mid \text{Ad}(k)|_{\overline{\mathcal{C}^{\Delta_\infty}}} = \text{id}\}$. Since $Z_K^{H_\infty} = Z_K^{\Delta_\infty}$ ([HKT]), for any accumulation point $\tilde{k} \in K$ of $\{k_n\}_{n \in \mathbb{N}}$, $\text{Ad}(\tilde{k})|_{\overline{\mathcal{C}^{\Delta_\infty}}} = \text{Ad}(k_\infty)|_{\overline{\mathcal{C}^{\Delta_\infty}}}$. Thus, we have $\lim_{n \rightarrow \infty} \text{Ad}(k_n)|_{\overline{\mathcal{C}^{\Delta_\infty}}} = \text{Ad}(k_\infty)|_{\overline{\mathcal{C}^{\Delta_\infty}}}$. Therefore

$$\lim_{n \rightarrow \infty} \Phi(P_n) = \lim_{n \rightarrow \infty} \text{Ad}(k_n)f(H_n)A = \text{Ad}(k_\infty)f(H_\infty)A = \Phi(P_\infty)$$

Hence Φ is a retraction from \mathfrak{m} onto $C_{\text{Ad}(K)A}$. \square

Proposition 2. *Let $\Phi : \mathfrak{m} \rightarrow C_{\text{Ad}(K)A}$ be a retraction which constructed by Theorem 2. If $\Phi|_{\mathfrak{a} \setminus \Phi^{-1}(\{0\})}$ is C^1 , then $\Phi|_{\mathfrak{m} \setminus \Phi^{-1}(\{0\})}$ is C^1 . In this case Φ is area-nonincreasing if and only if $J(d\Phi)_x \leq 1$ holds for each $x \in \mathcal{C} \setminus \Phi^{-1}(\{0\})$.*

Proof. If Φ is C^1 at $H \in \overline{\mathcal{C}}$, then Φ is C^1 at $\text{Ad}(k)H$ for all $k \in K$. Thus we assume $H \in \overline{\mathcal{C}} \setminus \Phi^{-1}(\{0\})$. For $H \in \overline{\mathcal{C}} \setminus \Phi^{-1}(\{0\})$, we put $\Delta = \{\alpha \in F \mid \langle \alpha, H \rangle > 0\}$. Since $f(H) > 0$, we get $\Delta_0 \subset \Delta$ and $\mathcal{C}^{\Delta_0} \subset \overline{\mathcal{C}^\Delta}$. By Lemma 1, we have

$$\mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in \tilde{R}_+ \setminus \tilde{R}_0} \mathbb{R} \cdot T_\lambda.$$

Since $\Phi|_{\mathfrak{a} \setminus \Phi^{-1}(\{0\})}$ is C^1 , we consider only T_λ direction for each $\lambda \in \tilde{R}_+ \setminus \tilde{R}_0$. If $\langle \lambda, H \rangle = 0$, then $[T_\lambda, H] = \langle \lambda, H \rangle S_\lambda = 0$ from Lemma 1. Thus there exists $k \in Z_K^H = \{k \in K \mid \text{Ad}(k)H = H\}$ such that $\text{Ad}(k)T_\lambda \in \mathfrak{a}$. Therefore

$$\Phi(H + tT_\lambda) = \text{Ad}(k)^{-1}\Phi(\text{Ad}(k)(H + tT_\lambda)).$$

Since $\text{Ad}(k)(H + tT_\lambda) \in \mathfrak{a}$ and $\Phi|_{\mathfrak{a} \setminus \Phi^{-1}(\{0\})}$ is C^1 , we have the directional derivative of Φ along T_λ . If $\langle \lambda, H \rangle \neq 0$, then from Lemma 1 we have that $c(t) = \text{Ad}(\exp(-tS_\lambda/\langle \lambda, H \rangle))H$ is curve in \mathfrak{m} with $c(0) = H$ and $c'(0) = T_\lambda$. Thus

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \Phi(c(t)) &= \left. \frac{d}{dt} \right|_{t=0} \Phi \left(\text{Ad} \left(\exp \frac{-tS_\lambda}{\langle \lambda, H \rangle} \right) H \right) \\ &= \frac{[-S_\lambda, \phi(H)]}{\langle \lambda, H \rangle} = \frac{\langle \lambda, A \rangle}{\langle \lambda, H \rangle} f(H) T_\lambda. \end{aligned}$$

Therefore Φ is a differentiable retraction from \mathfrak{m} into $C_{\text{Ad}(K)A}$. Since $\Phi|_{\mathfrak{m} \setminus \Phi^{-1}(\{0\})}$ is C^1 , the mapping $\overline{\mathcal{C}} \setminus \Phi^{-1}(\{0\}) \rightarrow \mathbb{R}; x \mapsto J(d\Phi_x)$ is continuous. Hence, if $J(d\Phi_x) \leq 1$ ($x \in \mathcal{C} \setminus \Phi^{-1}(\{0\})$), then $J(d\Phi_x) \leq 1$ ($x \in \overline{\mathcal{C}} \setminus \Phi^{-1}(\{0\})$). \square

We will compute $J(d\Phi_x)$ of Φ in Theorem 2 for $x \in \mathcal{C} \setminus \Phi^{-1}(\{0\})$.

Proposition 3. *We denote $R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A \rangle = 0\}$.*

$$J(d\Phi_x) = \|(\text{grad} f)_x\| \prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)} \quad (x \in \mathcal{C} \setminus \Phi^{-1}(\{0\})).$$

Proof. From the proof of Proposition 2, we have

$$d\Phi_x(H) = df_x(H)A \quad (H \in \mathfrak{a}), \quad d\Phi_x(T_\lambda) = \frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) T_\lambda \quad (\lambda \in \tilde{R}_+ \setminus \tilde{R}_0)$$

for $x \in \mathcal{C} \setminus \Phi^{-1}(\{0\})$. Thus we get

$$d\Phi_x(\mathfrak{a}) \subset \mathbb{R}A \subset \mathfrak{a}, \quad d\Phi_x \left(\sum_{\mu \in R_+} \mathfrak{m}_\mu \right) \subset \sum_{\mu \in R_+} \mathfrak{m}_\mu.$$

Since \mathfrak{a} and $\sum_{\mu \in R_+} \mathfrak{m}_\mu$ are orthogonal, we have

$$J(d\Phi_x) = J(d\Phi_x|_{\mathfrak{a}}) \times J(d\Phi_x|_{\sum_{\mu \in R_+} \mathfrak{m}_\mu}).$$

We put $J_1(x) = J(d\Phi_x|_{\mathfrak{a}})$, $J_2(x) = J(d\Phi_x|_{\sum_{\mu \in R_+} \mathfrak{m}_\mu})$ and compute each of these.

$$\begin{aligned} J_1(x) &= \sup\{\|d\Phi_x(v)\| \mid v \in \mathfrak{a}, \|v\| = 1\} \\ &= \sup\{\langle (\text{grad} f)_x, v \rangle \mid v \in \mathfrak{a}, \|v\| = 1\} = \|(\text{grad} f)_x\|. \end{aligned}$$

Since $\ker(d\Phi_x|_{\sum_{\mu \in R_+} \mathfrak{m}_\mu}) = \sum_{\mu \in R_+^{\Delta_0}} \mathfrak{m}_\mu$, $\{T_\lambda \mid \lambda \in \tilde{R}_+, \langle \lambda, A \rangle > 0\}$ is an orthonormal basis of $\ker(d\Phi_x|_{\sum_{\mu \in R_+} \mathfrak{m}_\mu})^\perp = \sum_{\mu \in R_+ \setminus R_+^{\Delta_0}} \mathfrak{m}_\mu$. Hence

$$\begin{aligned} J_2(x) &= \left\| \bigwedge_{\lambda \in \tilde{R}_+, \langle \lambda, A \rangle > 0} d\Phi_x(T_\lambda) \right\| = \left\| \bigwedge_{\lambda \in \tilde{R}_+, \langle \lambda, A \rangle > 0} \frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) T_\lambda \right\| \\ &= \prod_{\lambda \in \tilde{R}_+, \langle \lambda, A \rangle > 0} \frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) = \prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)}. \end{aligned}$$

Therefore we get

$$J(d\Phi)_x = J_1(x)J_2(x) = \|(\text{grad} f)_x\| \prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)}.$$

□

4. EXAMPLE OF AREA-MINIMIZING CONES OVER R-SPACES

Using Theorem 2, Proposition 2 and Proposition 3, we investigate area-minimizing properties of cones over R-spaces. First we consider cones over isolated orbits of s -representations of irreducible symmetric pairs of compact type of rank two. Principal orbits of these representations are homogeneous hypersurfaces in the sphere. The area-minimizing properties of the cones over homogeneous minimal hypersurfaces were investigated in [Hs] and [L].

We shall follow the notations of root systems in [Bo]. Partly we used Maxima¹ for algebraic computations.

4.1. Type A_2 .

$$\mathfrak{a} = \{\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \mid \xi_1 + \xi_2 + \xi_3 = 0\},$$

$$F = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3\}.$$

Then we have $R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. For $\lambda \in R_+$, we put $m = m(\lambda)$. We have

$$H_{\alpha_1} = \frac{1}{3}(2e_1 - e_2 - e_3), \quad H_{\alpha_2} = \frac{1}{3}(e_1 + e_2 - 2e_3).$$

We put

$$A_1 = \frac{H_{\alpha_1}}{\|H_{\alpha_1}\|} = \frac{1}{\sqrt{6}}(2e_1 - e_2 - e_3), \quad A_2 = \frac{H_{\alpha_2}}{\|H_{\alpha_2}\|} = \frac{1}{\sqrt{6}}(e_1 + e_2 - 2e_3).$$

Since $\text{Ad}(K)A_1$ and $\text{Ad}(K)A_2$ are isometric, we consider only the cone over $\text{Ad}(K)A_1$.

¹<http://maxima.sourceforge.net/>

4.1.1. *Cones over $\text{Ad}(K)A_1$.* We put $\Delta_0 = \{\alpha_1\}$ then $R_+^{\Delta_0} = \{\alpha_2\}$. For $x = x_1H_{\alpha_1} + x_2H_{\alpha_2} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{\frac{2}{3}} \left(\langle \alpha_1, x \rangle^2 \left\langle \alpha_1 + \frac{3}{2}\alpha_2, x \right\rangle \right)^{\frac{1}{3}} = \sqrt{\frac{2}{3}} \left(x_1^2 \left(x_1 + \frac{3}{2}x_2 \right) \right)^{\frac{1}{3}}.$$

Since

- (1) $f(tA_1) = \sqrt{\frac{2}{3}} \left(\left(\sqrt{\frac{3}{2}}t \right)^3 \right)^{\frac{1}{3}} = t,$
- (2) for each $\Delta \subset F$, if $\Delta_0 \not\subset \Delta$, then $f|_{\mathcal{C}^\Delta} = 0,$

we can apply Theorem 2 to this case. It is clear that $\Phi|_{\mathfrak{a} \setminus \Phi^{-1}(\{0\})}$ is C^1 . Thus Φ is a differentiable retraction by Proposition 2. Since

$$\begin{aligned} \frac{\partial f}{\partial x_1}(x) &= \sqrt{\frac{2}{3}} \left(x_1^2 \left(x_1 + \frac{3}{2}x_2 \right) \right)^{-\frac{2}{3}} (x_1^2 + x_1x_2), \\ \frac{\partial f}{\partial x_2}(x) &= \sqrt{\frac{2}{3}} \left(x_1^2 \left(x_1 + \frac{3}{2}x_2 \right) \right)^{-\frac{2}{3}} \frac{x_1^2}{2}, \end{aligned}$$

we get

$$J_1(x) = \|(\text{grad} f)_x\| = \sqrt{\frac{2}{3}} \left(x_1^2 \left(x_1 + \frac{3}{2}x_2 \right) \right)^{-\frac{2}{3}} \sqrt{\frac{3}{2}x_1^4 + 3x_1^3x_2 + 2x_1^2x_2^2}.$$

On the other hand,

$$J_2(x) = \left(\frac{\langle \alpha_1, A_1 \rangle}{\langle \alpha_1, x \rangle} f(x) \right)^m \left(\frac{\langle \alpha_1 + \alpha_2, A_1 \rangle}{\langle \alpha_1 + \alpha_2, x \rangle} f(x) \right)^m = \left(\frac{\left(x_1 \left(x_1 + \frac{3}{2}x_2 \right)^2 \right)^{\frac{1}{3}}}{x_1 + x_2} \right)^m.$$

Then

$$(x_1 + x_2)^3 - x_1 \left(x_1 + \frac{3}{2}x_2 \right)^2 = \frac{3}{4}x_1x_2^2 + x_2^3 \geq 0,$$

thus

$$\left(\frac{\left(x_1 \left(x_1 + \frac{3}{2}x_2 \right)^2 \right)^{\frac{1}{3}}}{x_1 + x_2} \right) \leq 1.$$

We put

$$D = J_1(x) \times \left(\frac{\left(x_1 \left(x_1 + \frac{3}{2}x_2 \right)^2 \right)^{\frac{1}{3}}}{x_1 + x_2} \right)^2 = \left(\frac{(3x_1^2 + 6x_1x_2 + 4x_2^2)^3 x_1^2 (2x_1 + 3x_2)^4}{3^3 2^4 (x_1 + x_2)^{12}} \right)^{\frac{1}{6}}.$$

Since

$$J(d\Phi)_x = D \times \left(\frac{\left(x_1 \left(x_1 + \frac{3}{2}x_2 \right)^2 \right)^{\frac{1}{3}}}{x_1 + x_2} \right)^{m-2},$$

if $D \leq 1$, then $J(d\Phi_x) \leq 1$ for $m \geq 2$. Since

$$\begin{aligned} & 3^3 2^4 (x_1 + x_2)^{12} - (3x_1^2 + 6x_1x_2 + 4x_2^2)^3 x_1^2 (2x_1 + 3x_2)^4 \\ &= 216x_1^{10}x_2^2 + 2376x_1^9x_2^3 + 11925x_1^8x_2^4 + 35838x_1^7x_2^5 + 71120x_1^6x_2^6 \\ & \quad + 96888x_1^5x_2^7 + 91152x_1^4x_2^8 + 57888x_1^3x_2^9 + 23328x_1^2x_2^{10} + 5184x_1x_2^{11} + 432x_2^{12} \\ & \geq 0, \end{aligned}$$

we have $D \leq 1$. Therefore, cones over $\text{Ad}(K)A_1$ are area-minimizing for $m \geq 2$.

4.2. Types B_2 , BC_2 and C_2 . Types C_2 and B_2 are isomorphic, thus it suffices to compute the type B_2 case. Moreover setting the multiplicity of long roots to zero, the set of restricted roots of type BC_2 reduces to that of type B_2 . We have

$$F = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\},$$

$$R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 2\alpha_2, 2\alpha_2\},$$

$$H_{\alpha_1} = e_1, \quad H_{\alpha_2} = e_1 + e_2,$$

and put

$$m(\alpha_1) = m_1, \quad m(\alpha_2) = m_2, \quad m(2\alpha_2) = m_3.$$

4.2.1. Cones over $\text{Ad}(K)A_1$. We put $\Delta_0 = \{\alpha_1\}$, then we have

$$A_1 = \frac{H_{\alpha_1}}{\|H_{\alpha_1}\|} = e_1,$$

and

$$R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A_1 \rangle = 0\} = \{\alpha_2, 2\alpha_2\}.$$

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{\langle \alpha_1, x \rangle \langle \alpha_1 + 2\alpha_2, x \rangle} = \sqrt{x_1(x_1 + 2x_2)}.$$

Then we can show that f satisfies the condition of Theorem 2 and Φ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m_2 + m_3 \geq 2$.

Therefore, cones over $\text{Ad}(K)A_1$ are area-minimizing for $m_2 + m_3 \geq 2$.

4.2.2. Cones over $\text{Ad}(K)A_2$. We put $\Delta_0 = \{\alpha_2\}$, then we have

$$A_2 = \frac{H_{\alpha_2}}{\|H_{\alpha_2}\|} = \frac{e_1 + e_2}{\sqrt{2}}$$

and

$$R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A_2 \rangle = 0\} = \{\alpha_1\}.$$

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{2} \left(\langle \alpha_2, x \rangle^2 \left\langle \frac{3}{2}\alpha_1 + \alpha_2, x \right\rangle \right)^{\frac{1}{3}} = \sqrt{2} \left(x_2^2 \left(\frac{3}{2}x_1 + x_2 \right) \right)^{\frac{1}{3}}.$$

Then we can show that f satisfies the condition of Theorem 2 and Φ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m_2 + m_3 \geq 2$.

Therefore, cones over $\text{Ad}(K)A_2$ are area-minimizing for $m_2 + m_3 \geq 2$.

4.3. Type G_2 . We have

$$F = \{\alpha_1, \alpha_2\},$$

$$R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\},$$

$$\langle \alpha_1, \alpha_1 \rangle = 1, \quad \langle \alpha_1, \alpha_2 \rangle = -\frac{3}{2}, \quad \langle \alpha_2, \alpha_2 \rangle = 3,$$

$$H_{\alpha_1} = 4\alpha_1 + 2\alpha_2, \quad H_{\alpha_2} = \frac{2}{3}(3\alpha_1 + 2\alpha_2),$$

and put

$$m = m(\alpha_1) = m(\alpha_2).$$

4.3.1. *Cones over $\text{Ad}(K)A_1$.* We put $\Delta_0 = \{\alpha_1\}$ then we have

$$A_1 = \frac{H_{\alpha_1}}{\|H_{\alpha_1}\|}$$

and

$$R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A_1 \rangle = 0\} = \{\alpha_2\}.$$

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{4\langle \alpha_1, x \rangle \langle \alpha_1 + \alpha_2, x \rangle} = \sqrt{4x_1(x_1 + x_2)}.$$

Then we can show that f satisfies the condition of Theorem 2 and Φ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m \geq 2$.

Therefore cones over $\text{Ad}(K)A_1$ are area-minimizing for $m \geq 2$.

4.3.2. *Cones over $\text{Ad}(K)A_2$.* We put $\Delta_0 = \{\alpha_2\}$ then we have

$$A_2 = \frac{H_{\alpha_2}}{\|H_{\alpha_2}\|},$$

and

$$R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A_1 \rangle = 0\} = \{\alpha_1\}.$$

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{\frac{4}{3}\langle \alpha_2, x \rangle \langle 3\alpha_1 + \alpha_2, x \rangle} = \sqrt{\frac{4}{3}x_2(3x_1 + x_2)}.$$

Then we can show that f satisfies the condition of Theorem 2 and Φ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m \geq 2$.

Therefore, cones over $\text{Ad}(K)A_2$ are area-minimizing for $m \geq 2$.

By the above computation, we get the following table of cones over isolated orbits of the s -representations of irreducible symmetric spaces of rank two.

type	symmetric pair	multiplicities	A_i	orbit	symm. or not	dim. of orbit and sphere	area-min.
A_2	$(\mathrm{SU}(3), \mathrm{SO}(3))$ $(\mathrm{SU}(3) \times \mathrm{SU}(3), \mathrm{SU}(3))$ $(\mathrm{SU}(6), \mathrm{Sp}(3))$ (E_6, F_4)	(1, 1)	A_1	$\mathbb{R}P^2$	symmetric	(2, 4)	
		(2, 2)	A_1	$\mathbb{C}P^2$	symmetric	(4, 7)	\bigcirc [Ke]
		(4, 4)	A_1	$\mathbb{H}P^2$	symmetric	(8, 13)	\bigcirc [Ka]
		(8, 8)	A_1	$\mathbb{O}P^2$	symmetric	(16, 25)	\bigcirc
B_2	$(\mathrm{SO}(5) \times \mathrm{SO}(5), \mathrm{SO}(5))$ $(\mathrm{SO}(5), \mathrm{SO}(2) \times \mathrm{SO}(3))$ $(\mathrm{SO}(4+n), \mathrm{SO}(2) \times \mathrm{SO}(2+n))$	(2, 2)	A_1	$\widetilde{G_2(\mathbb{R}^5)}$	symmetric	(6, 9)	\bigcirc [HKT]
		(1, 1)	A_2	$\mathrm{SO}(5)/\mathrm{U}(2)$	symmetric	(6, 9)	\bigcirc [Ke]
		(1, n)	A_1		symmetric	(3, 5)	
			A_2			(3, 5)	
			A_1			$(n+2, 2n+3)$	\bigcirc ($n \geq 2$) [HKT]
			A_2			$(2n+1, 2n+3)$	\bigcirc ($n \geq 2$)
C_2	$(\mathrm{Sp}(2), \mathrm{U}(2))$ $(\mathrm{Sp}(2) \times \mathrm{Sp}(2), \mathrm{Sp}(2))$ $(\mathrm{Sp}(4), \mathrm{Sp}(2) \times \mathrm{Sp}(2))$ $(\mathrm{SU}(4), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)))$ $(\mathrm{SO}(8), \mathrm{U}(4))$	(1, 1)	A_1	$\mathrm{U}(2)/\mathrm{O}(2)$	symmetric	(3, 5)	
		(2, 2)	A_2		symmetric	(3, 5)	
		(4, 3)	A_1	$\mathrm{Sp}(2)/\mathrm{U}(2)$	symmetric	(6, 9)	\bigcirc
		(2, 1)	A_2	$\mathrm{Sp}(2)$	symmetric	(6, 9)	\bigcirc
		(4, 1)	A_1	$\mathrm{U}(2)$	symmetric	(11, 15)	\bigcirc
			A_2	$\mathrm{U}(4)/(\mathrm{Sp}(1) \times \mathrm{U}(2))$	symmetric	(10, 15)	\bigcirc
			A_1	$\mathrm{U}(4)/\mathrm{Sp}(2)$	symmetric	(5, 7)	\bigcirc
			A_2		symmetric	(4, 7)	\bigcirc [L]
			A_1			(9, 11)	\bigcirc
			A_2			(6, 11)	\bigcirc

type	symmetric pair	multiplicities	A_i	orbit	symm. or not	dim. of orbit and sphere	area-min.
BC_2	$(SU(4+n), S(U(2) \times U(2+n)))$ $(SO(10), U(5))$ $(Sp(4+n), Sp(2) \times Sp(2+n))$ $(E_6, T^1 \cdot Spin(10))$	$(2, (2n, 1))$	A_1	$U(5)/(Sp(1) \times U(3))$ $U(5)/(Sp(2) \times U(1))$		$(2n+3, 4n+7)$	$\bigcirc (n \geq 1)$
		$(4, (4, 1))$	A_2			$(4n+4, 4n+7)$	$\bigcirc (n \geq 1)$
		$(4, (4n, 3))$	A_1			$(13, 19)$	\bigcirc
			A_2			$(14, 19)$	\bigcirc
		$(6, (8, 1))$	A_1			$(4n+11, 8n+15)$	$\bigcirc (n \geq 1)$
			A_2			$(8n+10, 8n+15)$	$\bigcirc (n \geq 1)$
			A_1			$(21, 31)$	\bigcirc
			A_2			$(24, 31)$	\bigcirc
G_2	$(G_2, SO(4))$ $(G_2 \times G_2, G_2)$	$(1, 1)$	A_1			$(5, 7)$	
			A_2			$(5, 7)$	
		$(2, 2)$	A_1			$(10, 13)$	\bigcirc
			A_2			$(10, 13)$	\bigcirc

4.4. **Type A_3 .** Theorem 2 can be applied to cones over minimal orbits, not only isolated orbits. We demonstrate the area-minimizing property for the cone over a minimal orbit, which is not an isolated orbit, of the s -representation of symmetric spaces of type A_3 .

$$\mathfrak{a} = \left\{ \sum_{i=1}^4 \xi_i e_i \mid \sum_{i=1}^4 \xi_i = 0 \right\},$$

$$F = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4\}.$$

Then $R_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ and for $\lambda \in R_+$, we put $m(\lambda) = m$. We have

$$H_{\alpha_1} = \frac{1}{4}(3e_1 - e_2 - e_3 - e_4), H_{\alpha_2} = \frac{1}{4}(2e_1 + 2e_2 - 2e_3 - 2e_4), H_{\alpha_3} = \frac{1}{4}(e_1 + e_2 + e_3 - 3e_4).$$

We put $\Delta_0 = \{\alpha_1, \alpha_3\}$, and we have

$$A = \frac{H_{\alpha_1} + H_{\alpha_3}}{\sqrt{2}} = \frac{e_1 - e_4}{\sqrt{2}}.$$

Then the orbit $\text{Ad}(K)A$ is a minimal submanifold of the sphere $S \subset \mathfrak{m}$. We get

$$R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A \rangle = 0\} = \{\alpha_2\}.$$

For $x = x_1 H_{\alpha_1} + x_2 H_{\alpha_2} + x_3 H_{\alpha_3} \in \overline{\mathcal{C}}$, we define

$$f(x) = \sqrt{2}(\langle \alpha_1, x \rangle \langle \alpha_3, x \rangle \langle \alpha_1 + \alpha_2, x \rangle \langle \alpha_2 + \alpha_3, x \rangle)^{\frac{1}{4}} = \sqrt{2}(x_1 x_3 (x_1 + x_2)(x_2 + x_3))^{\frac{1}{4}}.$$

Then we can show that f satisfies the condition of Theorem 2 and Φ is differentiable. Moreover $J(d\Phi_x) \leq 1$ holds for $m \geq 4$.

Therefore, cones over $\text{Ad}(K)A$ are area-minimizing for $m \geq 4$. The only symmetric pair which satisfies $m \geq 3$ is $(\text{SU}(6), \text{Sp}(3))$.

5. REDUCIBLE CASES

In this section, we consider cones over products of two R-spaces. Let (G_i, K_i) ($i = 1, 2$) be Riemannian symmetric pairs, and put $(G, K) = (G_1 \times G_2, K_1 \times K_2)$. We define the notation for (G_i, K_i) as follows. Let

$$\mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{m}_i \quad (i = 1, 2)$$

be the canonical decompositions of Lie algebras \mathfrak{g}_i of G_i . Take and fix a maximal abelian subspace \mathfrak{a}_i in \mathfrak{m}_i . We denote by R_i the restricted root system of $(\mathfrak{g}_i, \mathfrak{k}_i)$ with respect to \mathfrak{a}_i . We put the fundamental systems F_i of R_i by $F_i = \{\alpha_{i1}, \dots, \alpha_{il_i}\}$. R_{i+} is the set of positive roots in R_i . We set

$$\mathcal{C}_i = \{H \in \mathfrak{a}_i \mid \langle \alpha, H \rangle > 0 \ (\alpha \in F_i)\},$$

$$\mathcal{C}_i^{\Delta} = \{H \in \mathfrak{a}_i \mid \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \langle \beta, H \rangle = 0 \ (\beta \in F_i \setminus \Delta)\},$$

where $\Delta \subset F_i$. The direct sum of the s -representations of (G_i, K_i) is the s -representation of $(G, K) = (G_1 \times G_2, K_1 \times K_2)$. Then, we have

$$\overline{\mathcal{C}} = \overline{\mathcal{C}_1} \times \overline{\mathcal{C}_2}.$$

For $\Delta \subset F$, Δ is expressed as $\Delta = \Delta_1 \cup \Delta_2$ where $\Delta_i \subset F_i$ ($i = 1, 2$). By Theorem 1 for each Δ_i , there exists $A_i \in \overline{\mathcal{C}_i}$ such that $\text{Ad}(K_i)A_i$ is a minimal orbit of the s -representation of (G_i, K_i) . We put $k_i = \dim \text{Ad}(K_i)A_i$ and $k = k_1 + k_2$, then

$$A = \sqrt{\frac{k_1}{k}} A_1 + \sqrt{\frac{k_2}{k}} A_2 \in \overline{\mathcal{C}}$$

is a base point of a minimal orbit of the s -representation of (G, K) .

Theorem 3. *Let $\Delta_0 = \Delta_1 \cup \Delta_2$ ($\Delta_i \subset F_i$). We suppose that for the cone over $\text{Ad}(K_i)A_i$, there exists an area-nonincreasing retraction constructed by a function f_i on $\overline{\mathcal{C}}_i$ in Theorem 2, and that the retraction satisfies*

$$\prod_{\lambda \in R_{i+} \setminus R_{i+}^{\Delta_i}} \left(\frac{\langle \lambda, A_i \rangle}{\langle \lambda, x \rangle} f_i(x) \right)^{m(\lambda)} \leq 1 \quad (x \in \mathcal{C}_i).$$

If $\dim \text{Ad}(K_i)A_i \geq 3$, then there exists an area-nonincreasing retraction $\Phi : \mathfrak{m} \rightarrow C_{\text{Ad}(K)A}$ that constructed by some function f on $\overline{\mathcal{C}}$ in Theorem 2, and then the retraction satisfies

$$\prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)} \leq 1 \quad (x \in \mathcal{C}).$$

Proof. Let $k_i = \dim \text{Ad}(K_i)A_i$, $k = k_1 + k_2$ and put $a_i = \sqrt{k_i/k}$. $A = a_1 A_1 + a_2 A_2$ holds. For $x = (x_1, x_2) \in \overline{\mathcal{C}}_1 \times \overline{\mathcal{C}}_2 = \overline{\mathcal{C}}$ we define

$$f(x) = \begin{cases} \frac{f_1(x_1)f_2(x_2)}{a_2^3 f_1(x_1) + a_1^3 f_2(x_2)} & (f_1(x_1) \neq 0 \text{ or } f_2(x_2) \neq 0) \\ 0 & (f_1(x_1) = f_2(x_2) = 0) \end{cases}$$

We will show that f satisfies the conditions of Theorem 2. We can check easily $f(tA) = t$ for $t \geq 0$. For $\Delta \subset F$ with $\Delta_0 \not\subset \Delta$, using $\Delta'_i \subset F_i$ we can write $\Delta = \Delta'_1 \cup \Delta'_2$. Then $\Delta_i \not\subset \Delta'_i$ holds $i = 1$ or $i = 2$. Thus $f_1 = 0$ or $f_2 = 0$ holds on \mathcal{C}^Δ . Therefore $f|_{\mathcal{C}^\Delta} = 0$. Since $\Phi|_{\mathfrak{a} \setminus \{0\}}$ is C^1 , Φ is a differentiable retraction by Proposition 2. We calculate $J(d\Phi_x)$ for $x \in \mathcal{C} \setminus f^{-1}(\{0\})$. We put

$$J_1(x) = \|(\text{grad} f)_x\|, \quad J_2(x) = \prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)}$$

for $x = (x_1, x_2) = (x_1^1, \dots, x_1^{l_1}, x_2^1, \dots, x_2^{l_2}) \in \mathcal{C} \setminus f^{-1}(\{0\}) = \mathcal{C}_1 \times \mathcal{C}_2 \setminus f^{-1}(\{0\})$. Since

$$\begin{aligned} \frac{\partial f}{\partial x_1^j} &= \frac{\frac{\partial f_1}{\partial x_1^j} a_1^3 f_2(x_2)^2}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^2} \quad (j \in \{1, \dots, l_1\}), \\ \frac{\partial f}{\partial x_2^j} &= \frac{\frac{\partial f_2}{\partial x_2^j} a_2^3 f_1(x_1)^2}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^2} \quad (j \in \{1, \dots, l_2\}), \end{aligned}$$

we get

$$(\text{grad} f)_x = \frac{a_1^3 f_2(x_2)^2 (\text{grad} f_1)_{x_1} + a_2^3 f_1(x_1)^2 (\text{grad} f_2)_{x_2}}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^2}$$

and

$$J_1(x) = \|(\text{grad} f)_x\| = \frac{\sqrt{a_1^6 f_2(x_2)^4 \|(\text{grad} f_1)_{x_1}\|^2 + a_2^6 f_1(x_1)^4 \|(\text{grad} f_2)_{x_2}\|^2}}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^2}.$$

Since $R_+^{\Delta_0} = \{\lambda \in R_+ \mid \langle \lambda, A \rangle = 0\} = R_{1+}^{\Delta_1} \cup R_{2+}^{\Delta_2}$, we get

$$\begin{aligned} J_2(x) &= \prod_{\lambda \in R_+ \setminus R_+^{\Delta_0}} \left(\frac{\langle \lambda, A \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m(\lambda)} \\ &= \prod_{\lambda \in R_{1+} \setminus R_{1+}^{\Delta_1}} \left(\frac{\langle \lambda, a_1 A_1 \rangle}{\langle \lambda, x_1 \rangle} f(x) \right)^{m(\lambda)} \prod_{\mu \in R_{2+} \setminus R_{2+}^{\Delta_2}} \left(\frac{\langle \mu, a_2 A_2 \rangle}{\langle \mu, x_2 \rangle} f(x) \right)^{m(\mu)} \\ &= \prod_{\lambda \in R_{1+} \setminus R_{1+}^{\Delta_1}} \left(\frac{\langle \lambda, A_1 \rangle}{\langle \lambda, x_1 \rangle} f_1(x_1) \frac{a_1 f(x)}{f_1(x_1)} \right)^{m(\lambda)} \prod_{\mu \in R_{2+} \setminus R_{2+}^{\Delta_2}} \left(\frac{\langle \mu, A_2 \rangle}{\langle \mu, x_2 \rangle} f_2(x_2) \frac{a_2 f(x)}{f_2(x_2)} \right)^{m(\mu)}. \end{aligned}$$

Put

$$J_{2i}(x_i) = \prod_{\lambda \in R_{i+} \setminus R_{i+}^{\Delta_i}} \left(\frac{\langle \lambda, A_i \rangle}{\langle \lambda, x_i \rangle} f_i(x_i) \right)^{m(\lambda)}, \quad J_{1i}(x_i) = \|(\text{grad } f_i)_{x_i}\| \quad (i = 1, 2).$$

Note that $J_{2i}(x_i) \leq 1$, $J_{1i}(x_i)J_{2i}(x_i) \leq 1$ holds by assumption. Since

$$\sum_{\lambda \in R_{i+} \setminus R_{i+}^{\Delta_i}} m(\lambda) = \dim \text{Ad}(K_i)A_i = k_i,$$

we can write

$$J_2(x) = J_{21}(x_1)J_{22}(x_2) \left(\frac{a_1 f(x)}{f_1(x_1)} \right)^{k_1} \left(\frac{a_2 f(x)}{f_2(x_2)} \right)^{k_2}.$$

Since $J_{2i}(x_i) \leq 1$,

$$J_2(x) \leq \left(\frac{a_1 f(x)}{f_1(x_1)} \right)^{k_1} \left(\frac{a_2 f(x)}{f_2(x_2)} \right)^{k_2}.$$

We put

$$X_1 = \frac{f_2(x_2)}{a_2}, \quad X_2 = \frac{f_1(x_1)}{a_1}.$$

Then we have

$$\left(\frac{a_1 f(x)}{f_1(x_1)} \right)^{k_1} \left(\frac{a_2 f(x)}{f_2(x_2)} \right)^{k_2} = \frac{X_1^{k_1} X_2^{k_2}}{(a_1^2 X_1 + a_2^2 X_2)^k}.$$

For $X_1, X_2 > 0$, we define

$$\tilde{D}(X_1, X_2) = \frac{X_1^{k_1} X_2^{k_2}}{(a_1^2 X_1 + a_2^2 X_2)^k}.$$

If $\tilde{D} \leq 1$, then $J_2(x) \leq 1$. Thus we prove $\tilde{D} \leq 1$. Since $\tilde{D}(X_1, X_2) = \tilde{D}(tX_1, tX_2)$ ($t > 0$), in order to prove $\tilde{D} \leq 1$, we show $\tilde{D}|_P \leq 1$ where

$$P = \{(X_1, X_2) \in \mathbb{R}^2 \mid X_1, X_2 > 0, a_1^2 X_1 + a_2^2 X_2 = 1\}.$$

We have $\tilde{D}|_P = X_1^{k_1} X_2^{k_2}$ and $X_2 = \frac{1-a_1^2 X_1}{a_2^2}$. Since

$$\frac{d\tilde{D}|_P}{dX_1} = k_1 X_1^{k_1-1} X_2^{k_2} + X_1^{k_1} (-k_2 \frac{a_1^2}{a_2^2}) X_2^{k_2-1} = k_1 X_1^{k_1-1} X_2^{k_2-1} (X_2 - X_1),$$

a critical point of $\tilde{D}|_P$ is only $X_1 = 1$ in P . Further, we get

$$\tilde{D}|_P \rightarrow 0 \quad \text{as } X_1 \rightarrow 0 \text{ or } \frac{1}{a_1^2}.$$

Hence $\max\{\tilde{D}(X_1, X_2) \mid (X_1, X_2) \in P\} = \tilde{D}(1, 1) = 1$. Therefore

$$J_2(x) \leq 1.$$

Then we have

$$\begin{aligned}
 J(d\Phi)_x &= J_1(x)J_2(x) = \|(\text{grad } f)_x\|J_2(x) \\
 &= \frac{\sqrt{a_1^6 f_2(x_2)^4 J_{11}(x_1)^2 + a_2^6 f_1(x_1)^4 J_{12}(x_2)^2}}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^2} J_{21}(x_1)J_{22}(x_2) \frac{(a_1 f_2(x_2))^{k_1} (a_2 f_1(x_1))^{k_2}}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^k} \\
 &= \frac{\sqrt{a_1^6 f_2(x_2)^4 J_{11}(x_1)^2 J_{21}(x_1)^2 J_{22}(x_2)^2 + a_2^6 f_1(x_1)^4 J_{12}(x_2)^2 J_{21}(x_1)^2 J_{22}(x_2)^2}}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^{k+2}} \\
 &\quad \times (a_1 f_2(x_2))^{k_1} (a_2 f_1(x_1))^{k_2} \\
 &\leq \frac{\sqrt{a_1^6 f_2(x_2)^4 + a_2^6 f_1(x_1)^4} (a_1 f_2(x_2))^{k_1} (a_2 f_1(x_1))^{k_2}}{(a_2^3 f_1(x_1) + a_1^3 f_2(x_2))^{k+2}} \\
 &= \frac{\sqrt{a_1^2 X_1^4 + a_2^2 X_2^4} X_1^{k_1} X_2^{k_2}}{(a_1^2 X_1 + a_2^2 X_2)^{k+2}}.
 \end{aligned}$$

We define

$$D(X_1, X_2) = J(d\Phi_x)^2 = \frac{(a_1^2 X_1^4 + a_2^2 X_2^4) X_1^{2k_1} X_2^{2k_2}}{(a_1^2 X_1 + a_2^2 X_2)^{2k+4}}.$$

We have $D(tX_1, tX_2) = D(X_1, X_2)$ ($t > 0$). Similar to the above argument, we consider the maximum value of $D|_P$. Since

$$D|_P = (a_1^2 X_1^4 + a_2^2 X_2^4) X_1^{2k_1} X_2^{2k_2},$$

we get

$$\begin{aligned}
 \frac{dD|_P}{dX_1} &= 4 \left(a_1^2 X_1^3 - \frac{a_1^2}{a_2^2} a_2^2 X_2^3 \right) X_1^{2k_1} X_2^{2k_2} \\
 &\quad + (a_1^2 X_1^4 + a_2^2 X_2^4) \left(2k_1 X_1^{2k_1-1} X_2^{2k_2} - 2k_2 \frac{a_1^2}{a_2^2} X_1^{2k_1} X_2^{2k_2-1} \right) \\
 &= -2a_1^2 X_1^{2k_1-1} X_2^{2k_2-1} (X_1 - X_2) \\
 &\quad \times \{ ((k_1 - 3)X_1^4 + (k_2 - 3)X_2^4) + 3(X_1 - X_2)^4 + 10(X_1 - X_2)^2 \}.
 \end{aligned}$$

Hence, if $k_1 \geq 3$, $k_2 \geq 3$, then a critical point of $D|_P$ is only $X_1 = 1$ in P . Furthermore, we get

$$D|_P \rightarrow 0 \quad \text{as } X_1 \rightarrow 0 \text{ or } \frac{1}{a_1^2}.$$

Thus $\max\{D(X_1, X_2) \mid (X_1, X_2) \in P\} = D(1, 1) = 1$. Hence $D \leq 1$. This implies $J(d\Phi_x) \leq 1$. Therefore if $k_1 \geq 3$, $k_2 \geq 3$, Φ is area nonincreasing. \square

Remark 1. In 1969, Bombieri, DeGiorgi and Giusti [BDG] showed that the cone over $S^k \times S^k \subset S^{2k+1}$ ($k \geq 3$) is area-minimizing. On the other hand, Lawlor [L] proved that the cone over $S^{k_1} \times S^{k_2} \subset S^{k_1+k_2+1}$ are not area-minimizing when $k_1 + k_2 \leq 5$ or $k_1 = 1, k_2 = 5$. Hence, we need the condition $k_1 \geq 3, k_2 \geq 3$ in Theorem 3.

Remark 2. Area-nonincreasing retractions which we constructed in Section 4 satisfy the assumption of Theorem 3. Moreover, an area-nonincreasing retraction that is constructed using Theorem 3 satisfies the assumption of Theorem 3 again. Therefore, we can apply Theorem 3 inductively. This implies that the cone over a product of two or more R-spaces with “ \bigcirc ” in the table in Section 4 is area-minimizing.

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